1. 135(3). Antisymmetry Laws without the Constraint $\underline{A} = \underline{0}$.

On the u(1) level:

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}, \quad -(1)$$

$$\underline{B} = \underline{\nabla} \times \underline{A}. \quad -(2)$$

They are now constrained by:

$$\underline{\nabla}\phi = \frac{\partial \underline{A}}{\partial t} \quad -(3)$$

$$\frac{\partial A_i}{\partial x_j} = -\frac{\partial A_j}{\partial x_i} \quad -(4)$$

It follows that:

$$\underline{\nabla} \times \underline{\nabla}\phi = \underline{\nabla} \times \frac{\partial \underline{A}}{\partial t} = \underline{0}, \quad -(5)$$

so:

$$\underline{B} = \underline{\nabla} \times \underline{A} = \underline{0}. \quad -(6)$$

The antisymmetry constraints (3) and (4) cause the magnetic field to vanish. So u(1) electrodynamics is restricted by antisymmetry to electric fields only. In this note a potential is derived which is in accordance with the antisymmetry rule. The only situation in which the electric field does not vanish is defined by this potential:

$$\oint \underline{A} \cdot d\underline{l} = \int_S (\underline{\nabla} \times \underline{A}) \cdot \underline{n}\, dA \quad -(7)$$

$$= 0,$$

i.e by

$$\underline{\nabla} \times \underline{A} = \underline{0}, \quad -(8)$$

and by:

$$\frac{\partial A_X}{\partial Y} = -\frac{\partial A_Y}{\partial X} \quad \text{etc.} \qquad -(9)$$

Therefore:

$$\underline{A} = A_X \underline{i} + A_Y \underline{j} + A_Z \underline{k} \qquad -(10)$$

It also obeys the antisymmetry equation:

$$\frac{\partial \phi}{\partial Z} = \frac{\partial A_Z}{\partial t} \quad \text{etc.} \qquad -(11)$$

<u>Travelling Waves</u>

Attempt the solution:

$$\phi = \frac{\phi^{(0)}}{\sqrt{2}} \exp\left(i(\omega t - \kappa Z)\right) \qquad -(12)$$

Then:

$$\frac{\partial \phi}{\partial Z} = \frac{\partial A_Z}{\partial t} = -i\kappa\phi \qquad -(13)$$

$$A_Z = -i\kappa \int \phi \, dt \qquad -(14)$$

In general:

$$\underline{\nabla}\phi = -i\kappa\phi\left(\underline{i} + \underline{j} + \underline{k}\right) \qquad -(15)$$

$$\frac{\partial \underline{A}}{\partial t} = -\frac{i\omega}{c}\phi\left(\underline{i} + \underline{j} + \underline{k}\right) \qquad -(16)$$

$$\underline{E} = 2i\kappa\phi\left(\underline{i} + \underline{j} + \underline{k}\right) e^{i(\omega t - \kappa Z)} \qquad -(17)$$

$$\mathrm{Re}(\underline{E}) = -2\frac{\kappa}{c}\phi \sin(\omega t - \kappa Z)\left(\underline{i} + \underline{j} + \underline{k}\right) \qquad -(18)$$

In free space on the U(1) level:

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \qquad -(19)$$

$$\underline{\nabla} \times \underline{B} - \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} = \underline{0} \qquad -(20)$$

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(21)$$

$$\underline{\nabla} \cdot \underline{E} = 0 \quad -(22)$$

A solution of the type (18) does not obey:

$$\underline{\nabla} \times \underline{E} = \underline{0}, \quad \frac{\partial \underline{E}}{\partial t} = \underline{0}, \quad \underline{\nabla} \cdot \underline{E} = 0 \quad -(23)$$

so cannot be a free space solution. If for example:

$$\underline{E} = E_Z \, \underline{k}, \quad -(24)$$

$$E_Z = 2i\kappa\phi e^{i(\omega t - \kappa Z)} \quad -(25)$$

then

$$\underline{\nabla} \cdot \underline{E} = \frac{\partial E_Z}{\partial Z} = 2\kappa^2 e^{i(\omega t - \kappa Z)} \phi$$

$$= \rho / \epsilon_0 \quad -(26)$$

Therefore:

$$\boxed{\rho = 2\epsilon_0 \kappa^2 \phi e^{i(\omega t - \kappa Z)}} \quad -(27)$$

Similarly:

$$-\frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \mu_0 \underline{J} \quad -(28)$$

From eq. (25):

$$\frac{\partial E_Z}{\partial t} = -2\kappa\omega\phi e^{i(\omega t - \kappa Z)} \quad -(29)$$

So:

$$\boxed{J_Z = \frac{2\kappa\omega\phi}{\mu_0 c^2} e^{i(\omega t - \kappa Z)}} \quad -(30)$$

4) If this wave propagates at a velocity v:

$$\kappa = \frac{\omega}{v}. \quad -(31)$$

so:

$$\rho = 2\epsilon_0 \frac{\omega^2}{v^2} \phi e^{i(\omega t - \kappa z)} \quad -(32)$$

$$J_z = 2\epsilon_0 \frac{\omega^2}{v} \phi e^{i(\omega t - \kappa z)} \quad -(33)$$

On average:

$$\langle \rho \rangle = 0 \quad -(34)$$

$$\langle J_z \rangle = 0. \quad -(35)$$

and:

$$\langle \underline{E} \rangle = \underline{0}. \quad -(36)$$

The root mean squares are:

$$\langle \rho^2 \rangle^{1/2} = \epsilon_0 \left(\frac{\omega}{v}\right)^2 \phi \quad -(37)$$

$$\langle J_z^2 \rangle^{1/2} = \epsilon_0 \frac{\omega^2}{v} \phi \quad -(38)$$

$$\langle E_z^2 \rangle^{1/2} = \kappa \phi \quad -(39)$$

$$= \frac{\omega}{v} \phi \quad -(40)$$

<u>Coulomb Law</u>

In this case:

$$\phi = -\frac{e}{4\pi\epsilon_0 Z} \quad -(41)$$

$$\frac{\partial \phi}{\partial Z} = \frac{\partial A_z}{\partial t} = \frac{e}{4\pi\epsilon_0 Z^2} \quad -(42)$$

5), so:

$$A_Z = \int \frac{e}{4\pi\epsilon_0 Z^2}\, dt \quad -(43)$$

i.e.

$$A_Z(t) = \frac{et}{4\pi\epsilon_0 Z^2} + A_Z(0) \quad -(44)$$

$$\boxed{A_Z(t) - A_Z(0) = \frac{et}{4\pi\epsilon_0 Z^2}} \quad -(45)$$

Define:

$$v = \frac{Z}{t} \quad -(46)$$

then:

$$\boxed{\phi = -v A_Z} \quad -(47)$$

$$E_Z = -\frac{2e}{4\pi\epsilon_0 Z^2} \quad -(48)$$

ECE Level if $\underline{A} \neq \underline{0}$

The result is:

$$\boxed{\begin{aligned} \underline{E}^a &= \underline{E}_A^a + \phi^{(0)}\, \underline{\omega}_E \quad -(49) \\ \underline{B}^a &= \frac{\phi^{(0)}}{c}\, \underline{\omega}_B \quad -(50) \end{aligned}}$$

where $\underline{E}_A^a$ is due to a non-zero $\underline{A}$.