PROOF OF THE CARTAN EVANS IDENTITY

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ABSTRACT

The Cartan Evans identity is a new identity of differential geometry, and is the basis for the inhomogeneous field equation of Einstein Cartan Evans (ECE) theory. It is shown that it is a rigorous, self -checking, identity of differential geometry in the Riemannian manifold, and complete details of the proof are given for ease of reference.

Keywords: Cartan Evans identity, ECE theory, differential geometry.

1. INTRODUCTION

The Cartan Bianchi identity {1} is an identity of differential geometry in the Riemannian manifold, and is well known and proven rigorously. It has been shown in this series of papers {2-10} that the Cartan Bianchi identity is a rigorous identity of the Riemannian manifold in which Einstein Cartan Evans (ECE) theory is defined. In previous work it has been reduced to a self checking identity. The latter consists of the cyclic sum of three curvature tensors on one side of the identity, and the same cyclic sum of the definitions of the same curvature tensors on the other side. Cartan reduced this exact tensorial identity of Riemann geometry to the elegant format of Cartan's differential geometry in the Riemannian manifold {1}:

$$D \wedge T^a := d \wedge T^a + \omega^a_b \wedge T^b \qquad (1)$$

$$:= R^a_b \wedge q^b \qquad (2)$$

Here $d \wedge$ is the exterior derivative, T^a is the Cartan torsion form, ω^a_b is the Cartan spin connection, R^a_b is the Cartan curvature form, q^b is the Cartan tetrad form, and $\wedge$ is the Cartan wedge product {1-10}. The Cartan Bianchi identity is valid in the Riemannian manifold, as is well known, and Cartan geometry in the Riemannian manifold is well known to be equivalent to Riemann geometry, thought to be the geometry of natural philosophy (physics).

Note carefully that the base manifold indices in eq. (1) are suppressed, as is the convention {1} in differential geometry. The reason is that the base manifold indices are the same on either side of any equation of differential geometry. The Cartan a index is the index

of a Minkowski spacetime, tangential to the base manifold at a point P. Cartan geometry is a pure geometry, it is coordinate independent and generally covariant. General covariance is the basic requirement of the philosophy of general relativity, as is well known. In ECE theory the Cartan Bianchi identity becomes the homogeneous field equation {2-10}. The latter has been developed in comprehensive detail in the 136 previous papers of this series, in form, tensor and vector notation in the Riemannian manifold of physics. Since Cartan geometry applies to the tangential, Minkowski, spacetime indexed a , it is valid for any base manifold in which a tangent spacetime may be defined at point P. Cartan geometry is therefore valid in any orientable manifold of pure mathematics, not just the Riemannian manifold of physics. In a non-orientable manifold of pure mathematics, such as a Möbius type or chiral manifold, a tangent spacetime may also be defined, but is not necessarily uniquely defined, as is well known in pure mathematics. These exotic manifolds of pure mathematics appear however to be irrelevant to physics because there is no experimental evidence that show that they must be preferred to the Riemannian manifold. In the Riemannian manifold in which is defined, the Cartan Bianchi identity is always rigorously true. Similarly the tetrad postulate is always rigorously true in the Riemannian manifold in which the tetrad postulate is defined. The torsion and curvature of ECE theory are objects of the Riemannian manifold. Cartan's torsion form is a vector valued two-form {1 - 10} equivalent to the Riemannian torsion tensor. Cartan's curvature form is a tensor valued two-form equivalent to the Riemannian curvature tensor. ECE theory is based on experimental data.

In Section 2 the concept is introduced of the Hodge dual connection, and secondly the concept is introduced of the covariant derivative using the Hodge dual connection. Thereafter the proof of the Cartan Evans identity follows the proof of the Cartan Bianchi identity. In Section 2 it is given in all detail.

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2. DETAILED PROOF

In previous work {2-10} it has been shown that the Riemannian connection is antisymmetric in its lower two indices. This follows as soon as the Riemannian torsion is properly taken into account. In the obsolete physics known as "the standard model", the torsion was incorrectly assumed to be zero, and the connection incorrectly assumed in consequence to be symmetric in its lower two indices. The Riemannian torsion tensor is defined as:

$$T^{\kappa}_{\mu\nu} := \Gamma^{\kappa}_{\mu\nu} - \Gamma^{\kappa}_{\nu\mu} \tag{2}$$

and is antisymmetric in its lower two indices as is well known. This antisymmetry follows from the fundamental equation of Riemannian geometry {1 - 10}:

$$[D_{\mu}, D_{\nu}] V^{\rho} = R^{\rho}_{\sigma\mu\nu} V^{\sigma} - T^{\lambda}_{\mu\nu} D_{\lambda} V^{\rho} \tag{3}$$

where the antisymmetric commutator of covariant derivatives acts on the vector V^{ρ} in any spacetime of any dimension in the Riemannian manifold. In eq. (3) $R^{\rho}_{\sigma\mu\nu}$ is the curvature tensor in the Riemannian manifold. Thus, from eq. (2) in (3):

$$[D_{\mu}, D_{\nu}] V^{\rho} = - \Gamma^{\lambda}_{\mu\nu} D_{\lambda} V^{\rho} + \dots \tag{4}$$

and the connection must be antisymmetric in its lower two indices:

$$\Gamma^{\lambda}_{\mu\nu} = - \Gamma^{\lambda}_{\nu\mu} \tag{5}$$

simply because its indices are those of the antisymmetric commutator. If:

$$\mu = \nu \tag{6}$$

then the commutator and connection both vanish, as do the torsion and curvature tensors. The error of the obsolete physics was:

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu} \neq ? \, 0 \tag{7}$$

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and this was perpetuated uncritically for ninety years. This means that the cosmology of the standard model was meaningless, and has been replaced {1 - 10} by ECE cosmology based on torsion.

The antisymmetry of the connection as in Eq. (5) means that its Hodge dual in four dimensions is {1 - 10}:

$$\Lambda^{\lambda}_{\mu\nu} := \widetilde{\Gamma}^{\lambda}_{\mu\nu} = \tfrac{1}{2} \|g\|^{\frac{1}{2}} \epsilon^{\alpha\beta}_{\mu\nu} \Gamma^{\lambda}_{\alpha\beta} \tag{8}$$

where $\|g\|^{\frac{1}{2}}$ is the square root of the modulus of the determinant of the metric, a weighting factor, and where the totally antisymmetric unit tensor $\epsilon^{\alpha\beta}_{\mu\nu}$ is defined {1} in Minkowski spacetime, not the general spacetime. It is well known that the connection does not transform as a tensor under the general coordinate transformation, but the antisymmetry in its lower two indices means that its Hodge dual may be defined for each upper index of the connection as in eq. (8). The antisymmetry of the connection as in Eq. (8) is the basis for the Cartan Evans identity, a new and fundamental identity of differential geometry. In ECE theory {2 - 10} it becomes the inhomogeneous field equation. Note carefully that the torsion is a tensor, but the connection is not a tensor. The same is true of the Hodge duals of the torsion and connection.

With these fundamental definitions take the Hodge duals either side of Eq. (3) using:

$$[D_{\mu}, D_{\nu}]_{\mathsf{HD}} = \tfrac{1}{2} \|g\|^{\frac{1}{2}} \epsilon^{\alpha\beta}_{\mu\nu} [D_{\alpha}, D_{\beta}] \tag{9}$$

$$\widetilde{R}^{\rho}_{\sigma\mu\nu} = \tfrac{1}{2} \|g\|^{\frac{1}{2}} \epsilon^{\alpha\beta}_{\mu\nu} R^{\rho}_{\sigma\alpha\beta} \tag{10}$$

$$\widetilde{T}^{\lambda}_{\mu\nu} = \tfrac{1}{2} \|g\|^{\frac{1}{2}} \epsilon^{\alpha\beta}_{\mu\nu} T^{\lambda}_{\alpha\beta} \tag{11}$$

Thus:

$$[D_\alpha \, , D_\beta]_{\textbf{HD}} \, V^\rho = \widetilde{R}^\rho_{\alpha\beta}V^\sigma - \widetilde{T}^\lambda_{\alpha\beta}D_\lambda V^\rho \tag{12}$$

Re-label indices in Eq. (12) to give:

$$[D_\mu \, , D_\nu]_{\textbf{HD}} \, V^\rho = \widetilde{R}^\rho_{\sigma\mu\nu}V^\sigma - \widetilde{T}^\lambda_{\mu\nu}D_\lambda V^\rho \tag{13}$$

The left hand side of this equation is defined by {1 - 10}:

$$[D_\mu \, , D_\nu]_{\textbf{HD}} \, V^\rho : = D_\mu(D_\nu V^\rho) - D_\nu \, (D_\mu V^\rho) \tag{14}$$

where the covariant derivatives must be defined by the Hodge dual connection defined in Eq. (8):

$$D_\mu V^\rho = \partial_\mu V^\rho + \Lambda^\rho_{\mu\lambda}V^\lambda \tag{15}$$

$$D_\nu V^\rho = \partial_\nu V^\rho + \Lambda^\rho_{\nu\lambda}V^\lambda \tag{16}$$

Working out the algebra of Eq. (14) (see paper 99 on www.aias.us):

$$\widetilde{T}^\lambda_{\mu\nu} = \Lambda^\lambda_{\mu\nu} - \Lambda^\lambda_{\nu\mu} \tag{17}$$

$$\widetilde{R}^\lambda_{\mu\nu\rho} = \partial_\mu \Lambda^\lambda_{\nu\rho} - \partial_\nu \Lambda^\lambda_{\mu\rho} + \Lambda^\lambda_{\mu\nu} \, \Lambda^\sigma_{\nu\rho} - \Lambda^\lambda_{\nu\sigma} \, \Lambda^\lambda_{\mu\rho} \tag{18}$$

These are the Hodge dual torsion and curvature tensors of the Riemannian manifold.

Now prove the Cartan Evans identity as follows. The identity is:

$$d\wedge \widetilde{T}^a \, + \omega^a_b \wedge \widetilde{T}^b \, : = \widetilde{R}^a_b \wedge q^b$$

$$: = D\wedge \widetilde{T}^a \tag{19}$$

In tensorial notation in the Riemannian manifold Eq. (19) becomes {1 -10}:

$$D_\mu \, \widetilde{T}^a_{\nu\rho} + D_\rho \, \widetilde{T}^a_{\mu\nu} + D_\nu \, \widetilde{T}^a_{\rho\mu} \, : = \widetilde{R}^a_{\mu\nu\rho} + \widetilde{R}^a_{\rho\mu\nu} + \widetilde{R}^a_{\nu\rho\mu} \tag{20}$$

Now proceed to prove Eq. (20) in exactly the same way as proofs given of the Cartan Bianchi identity (for example in paper 102 of www.aias.us). It is required to prove that:

$$\partial_\mu \, \widetilde{T}^a_{\nu\rho} + \omega^a_{\mu b} \, \widetilde{T}^b_{\nu\rho} + \ldots = \widetilde{R}^\lambda_{\mu\nu\rho} q^a_\lambda + \ldots \tag{21}$$

By definition { 1 - 10}:

$$\widetilde{T}^a_{\mu\nu} = (\Lambda^\lambda_{\mu\nu} - \Lambda^\lambda_{\nu\mu}) \, q^a_\lambda \tag{22}$$

so Eq. (21) is:

$$\partial_\mu ((\, \Lambda^\lambda_{\nu\rho} - \Lambda^\lambda_{\rho\nu} \,) \, q^a_\lambda \,) + \omega^a_{\mu b} (\, \Lambda^\lambda_{\nu\rho} - \Lambda^\lambda_{\rho\nu} \,) \, q^b_\lambda + \ldots$$

$$:= \widetilde{R}^\lambda_{\mu\nu\rho} q^a_\lambda + \ldots \tag{23}$$

The Leibniz rule gives:

$$\partial_\mu ((\, \Lambda^\lambda_{\nu\rho} - \Lambda^\lambda_{\rho\nu} \,) \, q^a_\lambda \,) = q^a_\lambda \partial_\mu (\, \Lambda^\lambda_{\nu\rho} - \Lambda^\lambda_{\rho\nu} \,)$$

$$+ (\, \Lambda^\lambda_{\nu\rho} - \Lambda^\lambda_{\rho\nu} \,) \, \partial_\mu q^a_\lambda \tag{24}$$

So Eq. (23) becomes:

$$(\partial_\mu \Lambda^\lambda_{\nu\rho} - \partial_\mu \, \Lambda^\lambda_{\rho\nu}) \, q^a_\lambda + (\, \partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda) \, (\, \Lambda^\lambda_{\nu\rho} - \Lambda^\lambda_{\rho\nu})$$

$$+ \ldots \quad := \widetilde{R}^\lambda_{\mu\nu\rho} q^a_\lambda + \ldots \tag{25}$$

Re-label summation indices to give:

$$(\partial_\mu \Lambda^\lambda_{\nu\rho} - \partial_\mu \, \Lambda^\lambda_{\rho\nu}) \, q^a_\lambda + (\, \partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma) \, (\, \Lambda^\sigma_{\nu\rho} - \Lambda^\sigma_{\rho\nu})$$

$$+ \ldots \quad := \widetilde{R}^\lambda_{\mu\nu\rho} q^a_\lambda + \ldots \tag{26}$$

Use the tetrad postulate with the Hodge dual connection defined in Eq. (8):

$$\partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma = \Lambda^\lambda_{\mu\sigma} \, q^a_\lambda \tag{27}$$

This tetrad postulate follows from Eqs. (8) and (19) and the tetrad postulate has been proven rigorously in many ways in previous work {2 - 10}, so:

$$(\partial_\mu \Lambda^\lambda_{\nu\rho} - \partial_\mu \Lambda^\lambda_{\rho\nu})\, q^a_\lambda + \Lambda^\lambda_{\mu\sigma}\, (\Lambda^\sigma_{\nu\rho} - \Lambda^\sigma_{\rho\nu})\, q^a_\lambda$$

$$+ \ldots \; := \; \widetilde{R}^\lambda_{\mu\nu\rho} q^a_\lambda + \ldots \tag{28}$$

A solution of Eq. (26) is:

$$\partial_\mu \Lambda^\lambda_{\nu\rho} - \partial_\mu \Lambda^\lambda_{\rho\nu} + \Lambda^\lambda_{\mu\sigma}\, (\Lambda^\sigma_{\nu\rho} - \Lambda^\sigma_{\rho\nu}) + \partial_\rho \Lambda^\lambda_{\mu\nu} - \partial_\rho \Lambda^\lambda_{\nu\mu} + \Lambda^\lambda_{\rho\sigma}\, (\Lambda^\sigma_{\mu\nu} - \Lambda^\sigma_{\nu\mu}) +$$

$$\partial_\nu \Lambda^\lambda_{\rho\mu} - \partial_\nu \Lambda^\lambda_{\mu\rho} + \Lambda^\lambda_{\nu\sigma}\, (\Lambda^\sigma_{\rho\mu} - \Lambda^\sigma_{\mu\rho}) \; := \; \widetilde{R}^\lambda_{\mu\nu\rho} + \widetilde{R}^\lambda_{\rho\mu\nu} + \widetilde{R}^\lambda_{\nu\rho\mu} \tag{29}$$

Rearrange terms on the left hand side of Eq. (29) to give an exact identity:

$$\widetilde{R}^\lambda_{\mu\nu\rho} + \widetilde{R}^\lambda_{\rho\mu\nu} + \widetilde{R}^\lambda_{\nu\rho\mu} = \partial_\mu \Lambda^\lambda_{\nu\rho} - \partial_\nu \Lambda^\lambda_{\mu\rho} + \Lambda^\lambda_{\mu\sigma} \Lambda^\sigma_{\nu\rho} - \Lambda^\lambda_{\nu\sigma}\, \Lambda^\sigma_{\mu\rho}$$

$$+ \partial_\rho \Lambda^\lambda_{\mu\nu} - \partial_\mu \Lambda^\lambda_{\rho\nu} + \Lambda^\lambda_{\rho\sigma} \Lambda^\sigma_{\mu\nu} - \Lambda^\lambda_{\mu\sigma}\, \Lambda^\sigma_{\rho\nu}$$

$$+ \partial_\nu \Lambda^\lambda_{\rho\mu} - \partial_\rho \Lambda^\lambda_{\nu\mu} + \Lambda^\lambda_{\nu\sigma} \Lambda^\sigma_{\rho\mu} - \Lambda^\lambda_{\rho\sigma}\, \Lambda^\sigma_{\nu\mu} \tag{30}$$

where by definition:

$$\widetilde{R}^\lambda_{\mu\nu\rho} = \partial_\mu \Lambda^\lambda_{\nu\rho} - \partial_\nu \Lambda^\lambda_{\mu\rho} + \Lambda^\lambda_{\mu\sigma} \Lambda^\sigma_{\nu\rho} - \Lambda^\lambda_{\nu\sigma}\, \Lambda^\sigma_{\mu\rho}$$

$$\widetilde{R}^\lambda_{\rho\mu\nu} = \partial_\rho \Lambda^\lambda_{\mu\nu} - \partial_\mu \Lambda^\lambda_{\rho\nu} + \Lambda^\lambda_{\rho\sigma} \Lambda^\sigma_{\mu\nu} - \Lambda^\lambda_{\mu\sigma}\, \Lambda^\sigma_{\rho\nu}$$

$$\widetilde{R}^\lambda_{\nu\rho\mu} = \partial_\nu \Lambda^\lambda_{\rho\mu} - \partial_\rho \Lambda^\lambda_{\nu\mu} + \Lambda^\lambda_{\nu\sigma} \Lambda^\sigma_{\rho\mu} - \Lambda^\lambda_{\rho\sigma}\, \Lambda^\sigma_{\nu\mu} \tag{31}$$

Quod erat demonstrandum.

It is seen that the Cartan Evans identity is based on the fundamental definition of the Hodge dual curvature, and adds three of them in cyclic permutation.

By using the definition:

$$\widetilde{T}^a_{\mu\nu} = q^a_\lambda\, \widetilde{T}^\lambda_{\mu\nu} \tag{32}$$

it follows that:

$$D_\mu \tilde{T}^a_{\nu\rho} = (D_\mu q^a_\kappa) \tilde{T}^\kappa_{\nu\rho} + q^a_\kappa D_\mu \tilde{T}^\kappa_{\nu\rho} \tag{33}$$

using the Leibniz rule. Use the tetrad postulate:

$$D_\mu q^a_\kappa = 0 \tag{34}$$

to find that:

$$D_\mu \tilde{T}^a_{\nu\rho} = q^a_\kappa D_\mu \tilde{T}^\kappa_{\nu\rho} \tag{35}$$

It follows that:

$$D_\mu \tilde{T}^\kappa_{\nu\rho} + D_\rho \tilde{T}^\kappa_{\mu\nu} + D_\nu \tilde{T}^\kappa_{\rho\mu} := \tilde{R}^\kappa_{\mu\nu\rho} + \tilde{R}^\kappa_{\rho\mu\nu} + \tilde{R}^\kappa_{\nu\rho\mu} \tag{36}$$

which may be rewritten as:

$$D_\mu T^{\kappa\mu\nu} := R^{\kappa\mu\nu}_\mu \tag{37}$$

The easiest way to see this is to take a particular example:

$$D_1 \tilde{T}^\kappa_{23} + D_3 \tilde{T}^\kappa_{12} + D_2 \tilde{T}^\kappa_{31} := \tilde{R}^\kappa_{123} + \tilde{R}^\kappa_{312} + \tilde{R}^\kappa_{231} \tag{38}$$

and take Hodge duals term by term to find:

$$D_1 T^{\kappa 01} + D_3 T^{\kappa 03} + D_2 T^{\kappa 02} := R^{\kappa 01}_1 + R^{\kappa 03}_3 + R^{\kappa 02}_2 \tag{39}$$

which is an example of Eq. (37), Q.E.D.

Eq. (37) is the most useful format of the Cartan Evans identity. In this format the Cartan Bianchi identity is {1 - 10}:

$$D_\mu \tilde{T}^{\kappa\mu\nu} := \tilde{R}^{\kappa\mu\nu}_\mu \tag{40}$$

The error (7) works its way through all of the obsolete and incorrect cosmology, which should be discarded by scholars. It has been shown by computer algebra (papers 93 onwards of www.aias.us) that all the metrics of the Einstein field equation in the presence of matter give the erroneous result:

$$T^{\kappa\mu\nu} = ? \, 0 \quad , \quad R_\mu^{\kappa\mu\nu2} \neq ? \, 0 \tag{41}$$

Finally, the covariant derivative in Eq. (37) is defined by the rule for taking a covariant derivative of a rank three tensor { 1 - 10}:

$$D_\sigma \tilde{T}^\kappa_{\mu\nu} = \partial_\sigma \, \tilde{T}^\kappa_{\mu\nu} + \Lambda^\kappa_{\sigma\lambda} \, \tilde{T}^\lambda_{\mu\nu} - \Lambda^\lambda_{\sigma\mu} \, \tilde{T}^\kappa_{\lambda\nu} - \Lambda^\lambda_{\sigma\nu} \, \tilde{T}^\kappa_{\mu\lambda} \tag{42}$$

(see reference (1) and papers 50, 100, 102 and 109 for example on www.aias.us). Eq. (42) uses the Λ connection defined in Eq. (8). Similarly, Eq. (37) uses the Λ connection and Eq. (40) uses the Γ connection. The covariant derivative in Eq. (19) is defined by the covariant derivative of Cartan geometry {1 - 10}:

$$D \wedge \tilde{T}^a := d \wedge \tilde{T}^a + \omega^a_b \wedge \tilde{T}^b \tag{43}$$

where the spin connection must be defined in terms of the Λ connection by the tetrad postulate with Λ connection:

$$D_\mu \, q^a_\sigma = \partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma - \Lambda^\lambda_{\mu\sigma} q^a_\lambda = 0 \tag{44}$$

One of the novel inferences of the Cartan Evans identity is that there is a Hodge dual connection in the Riemannian manifold in four dimensions. This is a basic discovery, and may be developed in pure mathematics using any type of manifold. However that development is not of interest to physics by Ockham's Razor, and the need to test a theory against experimental data.

ACKNOWLEDGMENTS

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