3(1): Fundamental Development of Dynamics from Differential Geometry.

The basic hypothesis is that the position vector $\underset{\sim}{r}$ may be expressed as:

$$\underset{\sim}{r} = \underset{\sim}{r}^{(1)} + \underset{\sim}{r}^{(2)} + \underset{\sim}{r}^{(3)} \quad — (1)$$

so that in general:

$$r^a_\mu = r q^a_\mu \quad — (2)$$

where q^a_μ has the properties of the Cartan tetrad. Therefore

$$V^a = q^a_\mu V^\mu \quad — (3)$$

In three dimensions:

$$\left.\begin{aligned} a &= (1), (2), (3) \\ \mu &= 1, 2, 3 \end{aligned}\right\} — (4)$$

The a index denotes the complex circular basis:

$$\left.\begin{aligned} \underline{e}^{(1)} &= \frac{1}{\sqrt{2}}(\underline{i} - i\underline{j}), \\ \underline{e}^{(2)} &= \frac{1}{\sqrt{2}}(\underline{i} + i\underline{j}) \\ \underline{e}^{(3)} &= \underline{k} \end{aligned}\right\} — (5)$$

Thus:

$$V_x^{(1)}, \ldots, V_Z^{(3)}$$

are elements of V^a_μ. If, for example, $\underset{\sim}{r}$ denotes a circularly polarized plane wave, then:

$$\left.\begin{aligned} \underline{r}^{(1)} &= \frac{r^{(1)}}{\sqrt{2}}(\underline{i} - i\underline{j}) e^{i(\omega t - \kappa Z)} = \underline{r}^{(2)*}, \\ \underline{r}^{(3)} &= r^{(3)} \underline{k} \end{aligned}\right\} — (6)$$

The existence of q^a_μ comes from the fact that the position vector $\underset{\sim}{r}$ may be expressed in two ways:

$$\underline{r} = r^{(1)}\underline{e}^{(1)} + r^{(2)}\underline{e}^{(2)} + r^{(3)}\underline{e}^{(3)}$$
$$= r_x \underline{i} + r_y \underline{j} + r_z \underline{k} \quad - (7)$$

This is an example in three dimensions of the tetrad postulate, the complete vector field $\underline{r}$ is the same. The elements $q^{(1)}_x, \ldots, q^{(3)}_Z$ are therefore fundamental elements of differential geometry.

The other fundamental forms of differential geometry are the torsion and curvature. The torsion is defined by the tetrad:

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b \quad - (8)$$

and the spin connection $\omega^a{}_b$. The right hand side of eq. (8) is the exterior covariant derivative:

$$T^a = D \wedge q^a \quad - (9)$$

In differential geometry this is the most fundamental type of derivative. It produces the fundamental T^a from the fundamental q^a. If $D \wedge$ is applied to T it produces a curvature three-form $R^a{}_b \wedge q^b$:

$$D \wedge T^a := R^a{}_b \wedge q^b \quad (10)$$

where

$$R^a{}_b = D \wedge \omega^a{}_b \quad (11)$$

<u>Hypothesis</u>

The velocity in dynamics is obtained from the position tetrad through the $D \wedge$ operator.

This is a hypothesis of general relativity because the basics of dynamics are being derived from the basics of differential geometry.

3)

It is convenient to define:

$$\boxed{V^a = c\, D \wedge r^a} \quad (12)$$

where c is the vacuum speed of light. In tensor notation:

$$V^a_{\mu\nu} = c\left(\partial_\mu r^a_\nu - \partial_\nu r^a_\mu + \omega^a_{\mu b} r^b_\nu - \omega^a_{\nu b} r^b_\mu\right) \quad (13)$$

The velocity is therefore a vector valued two form:

$$V^a_{\mu\nu} = -V^a_{\nu\mu} \quad (14)$$

In four dimensional spacetime the a index is:

$$a = (0), (1), (2), (3) \quad (15)$$

and

$$\mu = 0, 1, 2, 3 \quad (16)$$

By definition:

$$V^a_{\mu\nu} = \begin{bmatrix} 0 & -V^a_x & -V^a_y & -V^a_z \\ V^a_x & 0 & -W^a_z & W^a_y \\ V^a_y & W^a_z & 0 & -W^a_x \\ V^a_z & -W^a_y & W^a_x & 0 \end{bmatrix} \quad (17)$$

$$r^a_\mu = \left(r^a_0, -\underline{r}^a\right) \quad (18)$$

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \quad (19)$$

$$\omega^a_{\mu b} = \left(\omega^a_{0b}, -\underline{\omega}^a_b\right) \quad (20)$$

With these definitions it is possible to write eq. (13) as two vector equations. It is seen that the vector valued two-form $V^a_{\mu\nu}$ defines the space like components of the vector $\underline{V}^a$ and vector $\underline{W}^a$.

as follows:

$$\left.\begin{array}{ll} V^a_x = -V^a_{01}, & V^a_{12} = -W^a_z, \\ V^a_y = -V^a_{02}, & V^a_{13} = W^a_y, \\ V^a_z = -V^a_{03}, & V^a_{23} = -W^a_x. \end{array}\right\} \quad (21)$$

Since these are space-like, then in eq. (17):

$$a = (1),\ (2),\ (3). \quad (22)$$

Now split eq. (13) into an orbital and spin part.

<u>Orbital</u> <u>Velocity</u>

$$\left.\begin{array}{l} V^a_{01} = c\left(\partial_0 r^a_1 - \partial_1 r^a_0 + \omega^a_{0b} r^b_1 - \omega^a_{1b} r^b_0\right) \\ V^a_{02} = c\left(\partial_0 r^a_2 - \partial_2 r^a_0 + \omega^a_{0b} r^b_2 - \omega^a_{2b} r^b_0\right) \\ V^a_{03} = c\left(\partial_0 r^a_3 - \partial_3 r^a_0 + \omega^a_{0b} r^b_3 - \omega^a_{3b} r^b_0\right) \end{array}\right\} \quad (23)$$

The orbital velocity is:

$$\underline{V}^a = V^a_{01}\,\underline{i} + V^a_{02}\,\underline{j} + V^a_{03}\,\underline{k}. \quad (24)$$

In vector component notation eq. (23) is

$$-V^a_x = c\left(-\frac{1}{c}\frac{\partial}{\partial t} r^a_x - \frac{\partial r^a_0}{\partial X} - \omega^a_{0b} r^b_x + \omega^a_{xb} r^b_0\right) \quad (25)$$

and so on. So:

$$\boxed{\underline{V}^a = \frac{\partial \underline{r}^a}{\partial t} + c\,\underline{\nabla} r^a_0 + c\,\omega^a_{0b}\,\underline{r}^b - c\, r^b_0\,\underline{\omega}^a_b} \quad (26)$$

This is the most <u>general</u> expression for velocity.

Spin Velocity

$$
\left.
\begin{aligned}
W^a_{12} &= c\left(\partial_1 r^a_2 - \partial_2 r^a_1 + \omega^a_{1b} r^b_2 - \omega^a_{2b} r^b_1\right)\\
W^a_{31} &= c\left(\partial_3 r^a_1 - \partial_1 r^a_3 + \omega^a_{3b} r^b_1 - \omega^a_{1b} r^b_3\right)\\
W^a_{23} &= c\left(\partial_2 r^a_3 - \partial_3 r^a_2 + \omega^a_{2b} r^b_3 - \omega^a_{3b} r^b_2\right)
\end{aligned}
\right\} \quad -(27)
$$

The spin velocity is

$$
\underline{W}^a = W^a_{23}\,\underline{i} + W^a_{31}\,\underline{j} + W^a_{12}\,\underline{k} \quad -(28)
$$

In vector component notation eq (27) is:

$$
-W^a_Z = c\left(-\frac{\partial r^a_Y}{\partial X} + \frac{\partial r^a_X}{\partial Y} + \omega^a_{Xb} r^b_Y - \omega^a_{Yb} r^b_X\right) \quad -(29)
$$

and so on. Using the definition of the curl operator:

$$
\underline{\nabla} \times \underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial X & \partial/\partial Y & \partial/\partial Z \\ A_X & A_Y & A_Z \end{vmatrix} \quad -(30)
$$

$$
= \left(\frac{\partial A_Z}{\partial Y} - \frac{\partial_Y}{\partial Z}\right)\underline{i} - \left(\frac{\partial A_Z}{\partial X} - \frac{\partial A_X}{\partial Z}\right)\underline{j} + \left(\frac{\partial A_Y}{\partial X} - \frac{\partial A_X}{\partial Y}\right)\underline{k}
$$

it is seen that:

$$
\boxed{\underline{W}^a = c\left(\underline{\nabla} \times \underline{r}^a - \underline{\omega}^a_b \times \underline{r}^b\right)} \quad -(31)
$$

This is the most general expression for spin velocity, related to angular velocity.

6) It is seen that both $\underline{v}^a$ and $\underline{w}^a$ are vectors in space in three dimensions.

In four dimensional spacetime they are the space-like components of:

$$v^a_\mu = \left(v^a_0, -\underline{v}^a\right) \quad (32)$$

and

$$w^a_\mu = \left(w^a_0, -\underline{w}^a\right). \quad (33)$$

These are also defined by tetrads:

$$v^a_\mu = v\, q^a_\mu, \quad w^a_\mu = w\, q^a_\mu \quad (34)$$

and so there exist

$$\left.\begin{array}{l} v^{(0)}_\mu = \left(v^{(0)}_0, \underline{0}\right) \\ w^{(0)}_\mu = \left(w^{(0)}_0, \underline{0}\right) \end{array}\right\} \quad (35)$$

Acceleration

This is similarly defined by

$$\boxed{a^a = c\, D \wedge v^a} \quad (36)$$

and by

$$\boxed{\alpha^a = c\, D \wedge w^a} \quad (37)$$

Therefore there are four types of acceleration derivable from the two equations (36) and (37). Both a^a and α^a have orbital and spin parts.

7)

These are:

$$a^a_{orbital} = \frac{\partial \underline{v}^a}{\partial t} + c\underline{\nabla} v^a_0 + c\omega^a_{0b}\underline{v}^b - c v^b_0 \underline{\omega}^a_b \quad -(38)$$

$$a^a_{spin} = c\left(\underline{\nabla} \times \underline{v}^a - \underline{\omega}^a_b \times \underline{v}^b\right) \quad -(39)$$

$$\alpha^a_{orbital} = \frac{\partial \underline{w}^a}{\partial t} + c\underline{\nabla} w^a_0 + c\omega^a_{0b}\underline{w}^b - c w^b_0 \underline{\omega}^a_b \quad -(40)$$

$$\alpha^a_{spin} = c\left(\underline{\nabla} \times \underline{w}^a - \underline{\omega}^a_b \times \underline{w}^b\right) \quad -(41)$$

where:

$$\underline{v}^a = \frac{\partial \underline{r}^a}{\partial t} + c\underline{\nabla} r^a_0 + c\omega^a_{0b}\underline{r}^b - c r^b_0 \underline{\omega}^a_b \quad -(42)$$

and

$$\underline{w}^a = c\left(\underline{\nabla} \times \underline{r}^a - \underline{\omega}^a_b \times \underline{r}^b\right) \quad -(43)$$

In general there are forty eight types of acceleration, 16 from eq. (38), 16 from eq. (40), 8 from eq. (39) and 8 from eq. (43). These include the Newtonian acceleration due to inertial and gravitational mass, the Eulerian accelerations, and various other such as Coriolis and centripetal, and fundamental forces and torques. Some of these may not be known. The Coriolis acceleration is the second term in eq. (39) for example, but there is a new type of Coriolis acceleration in eq. (41):

$$\underline{\omega}^a_b \times \underline{w}^b = \underline{\omega}^a_b \times \left(\underline{\omega}^b_c \times \underline{r}^c\right) \quad -(44)$$

– the centripetal.