SIMPLIFIED PROOFS OF THE CARTAN STRUCTURE EQUATIONS

by

M. W. Evans,

Civil List Scientist

Alpha Institute for Advanced Studies,

(www.aias.us)

ABSTRACT

The two Cartan structure equations are proven straightforwardly through use of a simplified format for the tetrad postulate. In so doing a new general condition on Cartan's differential geometry is discovered and illustrated with respect to the tetrads of a propagating, circularly polarized, wave.

Keywords: Cartan structure equations, simplified tetrad postulate, new equation of Cartan's differential geometry.

1. INTRODUCTION

The first and second Cartan structure equations define the torsion and curvature forms of differential geometry and are equivalent to the Riemannian torsion and curvature (1-10}. They show the rigorous and complete internal self-consistency of Cartan geometry and illustrate the role played by the Cartan tetrad and the Cartan tetrad postulate. It is therefore important to be able to prove the structure equations as simply and as clearly as possible and to be able to prove their equivalence to nineteenth century Riemann geometry. The role of the a index of Cartan geometry is also clarified and well illustrated by these proofs, which are given in Section 2. In Section 3 the proof of the second structure equation is illustrated with reference to the tetrad of the propagating and circularly polarized wave. If a geometry is internally self-consistent it stands as such and may be used in the philosophy of general relativity as in ECE theory to give new physics and unify older concepts of physics.

2. PROOFS OF THE STRUCTURE EQUATIONS OF CARTAN

In the standard notation {1} of differential geometry the first Cartan structure equation is:

$$T^a = d \wedge q^a + \omega^a_b \wedge q^b \tag{1}$$

where T^a is the torsion form, a vector valued two-form, q^a is the tetrad form (a vector valued one-form), ω^a_b is the Cartan spin connection, and $d \wedge$ is the exterior derivative. In tensor notation this notation translates into:

$$T^a_{\mu\nu} = \partial_\mu\, q^a_\nu - \partial_\nu\, q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^b_{\nu b} q^b_\mu \tag{2}$$

The proof is now given of the equivalence of Eq. (2) and the Riemannian torsion:

$$T^\kappa_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu} \tag{3}$$

where $T^\kappa_{\mu\nu}$ is the Riemannian torsion and where $\Gamma^\kappa_{\mu\nu}$ is the Riemannian connection. The advantage of Eq. (2) over Eq. (3) is that Eq. (2) uses an index a that may be developed with any basis elements such as the Pauli matrices of Dirac theory. In Paper 141 of this series it was shown that the a index may be used to denote the complex circular basis, where it has a profound meaning in that it extends the Helmholtz Theorem.

The equivalence of Eqs. (2) and (3) originates in the fundamental property that the complete vector field is independent {1-10} of its coordinates and basis elements:

$$V = V^{\mu} e_{\mu} = V^{a} e_{a} \tag{4}$$

This is always the case in physics and general relativity. Without the property (4) a three dimensional vector field for example would be different in Cartesian or spherical polar coordinates. The property (4) leads to the equation:

$$D_{\mu} q^{a}_{\nu} = \partial_{\mu} q^{a}_{\nu} + \omega^{a}_{\mu b} q^{b}_{\nu} - \Gamma^{\lambda}_{\mu\nu} q^{a}_{\lambda} \tag{5}$$

This is confusingly and obscurely known as "the tetrad postulate"{1}. Eq.(5) is fundamentally true in physics and in almost all of mathematics, and is not a postulate. It follows from the property (4) {1-10}. Using the rules {1}:

$$\omega^{a}_{\mu\nu} = \omega^{a}_{\mu b} q^{b}_{\nu} \tag{6}$$

$$\Gamma^{\lambda}_{\mu\nu} q^{a}_{\lambda} = \Gamma^{a}_{\mu\nu} , \tag{7}$$

the tetrad postulate simplifies to:

$$\Gamma^{a}_{\mu\nu} = \partial_{\mu} q^{a}_{\nu} + \omega^{a}_{\mu\nu} \tag{8}$$

and the first Cartan structure equation simplifies to:

$$T^{a}_{\mu\nu} = \Gamma^{a}_{\mu\nu} - \Gamma^{a}_{\nu\mu} \tag{9}$$

It follows immediately that the Riemannian torsion is:

$$T^{\kappa}_{\mu\nu} = q^{\kappa}_{a} T^{a}_{\mu\nu} = \Gamma^{\kappa}_{\mu\nu} - \Gamma^{\kappa}_{\nu\mu} \tag{10}$$

Q.E.D.

The Cartan and Riemann torsions are both very fundamental concepts of geometry, in any space, of any dimensions. The Riemann torsion is always present in any space, and is defined by the commutator of covariant derivatives in any space of any dimension {1-10}:

$$[D_\mu, D_\nu] \; V^\rho = R^\rho_{\sigma\mu\nu} V^\sigma - T^\kappa_{\mu\nu} \, D_\kappa V^\rho \tag{11}$$

Here V^ρ is any vector, $R^\rho_{\sigma\mu\nu}$ is the Riemannian curvature and $D_\kappa V^\rho$ is the covariant derivative of V^ρ. From Eq. (11) it is obvious that the Riemannian connection must be antisymmetric:

$$\Gamma^\kappa_{\mu\nu} = - \Gamma^\kappa_{\nu\mu} \tag{12}$$

because the indices μ and ν are the same on both sides of the equation and because the commutator is antisymmetric by definition {1-10}. The connection can never be symmetric because the commutator can never be symmetric and at the same time non-zero.

The second Cartan structure equation {1-10} is:

$$R^a_b = d \wedge \omega^a_b + \omega^a_c \wedge \omega^c_b \tag{13}$$

where R^a_b is the curvature form (a tensor valued two-form). In tensor notation Eq. (13) is:

$$R^a_{b\mu\nu} = \partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} + \omega^a_{\mu c} \, \omega^c_{\nu b} - \omega^a_{\nu c} \, \omega^c_{\mu b} \tag{14}$$

and it is now proven that this equation is equivalent to the Riemann curvature:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \, \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \, \Gamma^\lambda_{\mu\sigma} \tag{15}$$

Start the proof by noting that:

$$\begin{aligned} \omega^a_{\mu b} &= q^\nu_b \omega^a_{\mu\nu} = \Gamma^a_{\mu b} - q^\nu_b \, \partial_\mu q^a_\nu \\ &= \Gamma^a_{\mu b} - \partial_\mu q^a_b \end{aligned} \tag{16}$$

Note that:

$$q^\nu_b \, \partial_\mu q^a_\nu = \partial_\mu q^a_b \tag{17}$$

because the tetrad multiplies a rank three mixed index tensorial quantity $\partial_\mu q^a_\nu$.

Therefore:

$$R^a_{b\mu\nu} = \partial_\mu \Gamma^a_{\nu b} - \partial_\nu \Gamma^a_{\mu b} + \Gamma^a_{\mu c}\, \Gamma^c_{\nu b} - \Gamma^a_{\nu c}\, \Gamma^c_{\mu b} - (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\, q^a_b$$

$$- \Gamma^c_{cb}\, \partial_\mu q^a_\nu - \Gamma^a_{cb}\, \partial_\nu q^c_\mu + \Gamma^c_{cb}\, \partial_\mu q^a_\nu + \Gamma^a_{cb}\, \partial_\nu q^c_\mu$$

$$= \partial_\mu \Gamma^a_{\nu b} - \partial_\nu \Gamma^a_{\mu b} + \Gamma^a_{\mu c}\, \Gamma^c_{\nu b} - \Gamma^a_{\nu c}\, \Gamma^c_{\mu b} \tag{18}$$

Finally use:

$$R^\rho_{\sigma\mu\nu} = q^\rho_a\, q^b_\sigma\, R^a_{b\mu\nu} \tag{19}$$

and

$$\Gamma^\rho_{\mu c}\, \Gamma^c_{\nu\sigma} = q^c_\lambda\, q^\lambda_c\, \Gamma^\rho_{\mu\lambda}\, \Gamma^\lambda_{\nu\sigma} = \Gamma^\rho_{\mu\lambda}\, \Gamma^\lambda_{\nu\sigma} \tag{20}$$

to obtain Eq. (15), Q.E.D.

This is a much simpler and clearer proof than one previously given in an appendix of Paper 15 of the ECE series.

By the rules of Cartan geometry it follows that:

$$q^a_b = q^a_\mu\, q^\mu_b \tag{21}$$

so using the Leibniz Theorem:

$$\partial_\nu q^a_b = q^\mu_b\, \partial_\nu q^a_\mu + q^a_\mu\, \partial_\nu\, q^\mu_b \tag{22}$$

The tetrad q^a_b in a Minkowski spacetime is unit diagonal:

$$q^a_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{23}$$

so:

$$\partial_\nu q^a_b = 0 \tag{24}$$

From Eqns. (17) and (22) it follows that:

$$q^a_\mu \partial_\nu q^\mu_b = 0 \tag{25}$$

and this is a new condition or constraint on Cartan geometry because the latter uses the index a to denote a Minkowski spacetime {1-10}. In ECE theory the potential is defined in terms of the tetrad {2-10}, so for example the electromagnetic potential is:

$$A_\mu^a = A^{(0)}\, q_\mu^a \tag{26}$$

Therefore:

$$A_\mu^a\, \partial_\nu\, A_b^\mu = 0 \tag{27}$$

and this is a new general constraint on the electromagnetic potential in ECE theory. Similarly, in gravitational theory:

$$\Phi_\mu^a\, \partial_\nu\, \Phi_b^\mu = 0 \tag{28}$$

where Φ_μ^a is the gravitational potential. Eq. (8) and (25) show that:

$$\Gamma_{\mu b}^a = \omega_{\mu b}^a \tag{29}$$

so the equivalence of the Cartan and Riemann curvatures follows immediately.

Note that the metric tensor is defined in general by:

$$x_\mu = g_{\mu\nu} x^\nu \tag{30}$$

where x^ν is the position four-vector:

$$x^\nu = (\, c\,t, X, Y, Z) \tag{31}$$

In Minkowski spacetime:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{32}$$

Eq. (30) can be expressed as:

$$x^\mu = g_\nu^\mu\, x^\nu \tag{33}$$

where:

$$g_\nu^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{34}$$

In Cartan's differential geometry the tetrad is defined in general by {1-10}:

$$V^a = q^a_\mu V^\mu \tag{35}$$

where V is any vector field. Eq (33) is a special case of Eq. (35):

$$x^a = q^a_b \, x^b \tag{36}$$

so it follows that:

$$q^a_b = g^a_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{37}$$

Q.E.D.

To exemplify Eq. (25) consider the following plane wave tetrads:

$$\boldsymbol{q}^{(1)} = \frac{1}{\sqrt{2}} (\mathbf{i} - i \, \mathbf{j}) \, e^{i\varphi} \tag{38}$$

and

$$\boldsymbol{q}^{(2)} = \frac{1}{\sqrt{2}} (\mathbf{i} + i \, \mathbf{j}) \, e^{-i\varphi} \tag{39}$$

Eq. (39) is the complex conjugate of Eq. (38) and φ is the phase of the plane wave. The complex circular basis is defined by:

$$\boldsymbol{q}^{(1)} \times \boldsymbol{q}^{(2)} = i \, \boldsymbol{q}^{(3)*} \tag{40}$$

in cyclic permutation.

Consider elements such as:

$$q^{(1)}_X = \frac{1}{\sqrt{2}} e^{i\varphi} \quad , \quad q^{(1)}_Y = \frac{-i}{\sqrt{2}} e^{i\varphi} \tag{41}$$

$$q^{(2)}_X = \frac{1}{\sqrt{2}} e^{-i\varphi} \, , \quad q^{(2)}_Y = \frac{i}{\sqrt{2}} e^{-i\varphi} \tag{42}$$

and use the rule:

$$q^a_\mu \, q^\mu_a = 1 \tag{43}$$

of Cartan geometry to find that:

$$q^{(1)}_X \, q^X_{(1)} + q^{(1)}_Y \, q^Y_{(1)} + q^{(2)}_X \, q^X_{(2)} + q^{(2)}_Y \, q^Y_{(2)} = 1 \tag{43a}$$

A solution of Eq. (43) is:

$$\mathrm{q}_{(1)}^{X} = \frac{1}{\sqrt{2}}\, e^{-i\varphi} \quad , \quad \mathrm{q}_{(1)}^{Y} = \frac{i}{\sqrt{2}}\, e^{-i\varphi} \quad ,$$

$$\mathrm{q}_{(2)}^{X} = \frac{1}{\sqrt{2}}\, e^{i\varphi} \quad , \quad \mathrm{q}_{(2)}^{Y} = \frac{-i}{\sqrt{2}}\, e^{i\varphi} \tag{44}$$

Consider an example of Eq. (25):

$$q_{\mu}^{a} \partial_{\nu} q_{b}^{\mu} = \mathrm{q}_{X}^{(1)} \partial_{\nu}\, \mathrm{q}_{(2)}^{X} + \mathrm{q}_{Y}^{(1)} \partial_{\nu}\, \mathrm{q}_{(2)}^{Y} \tag{45}$$

For:

$$\nu = 0 \tag{46}$$

it follows that:

$$\mathrm{q}_{X}^{(1)} \partial_{0}\, \mathrm{q}_{(2)}^{X} + \mathrm{q}_{Y}^{(1)} \partial_{0}\, \mathrm{q}_{(2)}^{Y} = 0 \tag{47}$$

For:

$$\nu = 3 \tag{48}$$

it follows that:

$$\mathrm{q}_{X}^{(1)} \partial_{3}\, \mathrm{q}_{(2)}^{X} + \mathrm{q}_{Y}^{(1)} \partial_{3}\, \mathrm{q}_{(2)}^{Y} = 0 \tag{49}$$

Consider a second example of Eq. (25):

$$q_{\mu}^{a} \partial_{\nu} q_{b}^{\mu} = \mathrm{q}_{X}^{(2)} \partial_{\nu}\, \mathrm{q}_{(1)}^{X} + \mathrm{q}_{Y}^{(2)} \partial_{\nu}\, \mathrm{q}_{(1)}^{Y} \tag{50}$$

and it follows that:

$$\mathrm{q}_{X}^{(2)} \partial_{0}\, \mathrm{q}_{(1)}^{X} + \mathrm{q}_{Y}^{(2)} \partial_{0}\, \mathrm{q}_{(1)}^{Y} = 0 \tag{51}$$

and:

$$\mathrm{q}_{X}^{(2)} \partial_{3}\, \mathrm{q}_{(1)}^{X} + \mathrm{q}_{Y}^{(2)} \partial_{3}\, \mathrm{q}_{(1)}^{Y} = 0 \tag{52}$$

So the general constraint (25) has been tested with the tetrads of the circularly polarized plane wave, and a self consistent result found. This procedure illustrates the importance of simplicity and clarity in mathematics and shows that within its own terms of definition, the two structure equations, Cartan's geometry is perfectly self consistent. Similarly, within its terms of reference, Euclid's geometry is perfectly self consistent, or Riemann's geometry is perfectly self consistent. It may be possible to have an ultra abstract geometry in which a vector field is not independent of its coordinate and basis elements, but that type of geometry merely adds abstraction and contravenes the most fundamental principle of general relativity, general covariance. There is no experimental evidence that such geometries have any role in nature. For example it may be possible to define manifolds other than that of Riemann, but again there is no evidence that these manifolds are relevant to natural philosophy (physics). The construction of such manifolds without any experimental evidence is contrary to the most fundamental rule of natural philosophy, Ockham's Razor. In other words they are just mathematics for the sake of mathematics. String theory falls into the same category, and any theory that contains unobservables. The twentieth century is pockmarked with such failed theories. ECE theory on the other hand has been tested experimentally in many ways {2-10} and is fundamentally very simple, being based directly on Cartan geometry. It should not surprise anyone that all the equations of physics come out of geometry using Ockham's Razor.

ACKNOWLEDGMENTS

The British Government is thanked for a Civil List Pension and Armorial bearings in recognition of contributions to Britain in science, and colleagues for many interesting discussions. The Alex Hill Group is thanked for typesetting.

REFERENCES

{1} S. P. Carroll, “Spacetime and Geometry: an Introduction to General Relativity” (Addison Wesley, New York, 2004), chapter 3.

{2} M. W. Evans, “Generally Covariant Unified Field Theory” (Abramis, 2005 onwards), in seven volumes to date, volume 7 with H. Eckardt, D. Lindstrom and F. Lichtenberg.

{3} L. Felker, “The Evans Equations of Unified Field Theory” (Abramis 2007).

{4} The ECE websites www.aias.us and www.atomicprecision.com

{5} K. Pendergast, “The Life of Myron Evans” (Abramis in press).

{6} M. W. Evans, H. Eckardt, S. Crothers and K. Pendergast, “Criticisms of the Einstein Field Equation” (Abramis in prep.).

{7} M. W. Evans (ed.), “Modern Nonlinear Optics” (Wiley 2001, second edition).

{8} M. W. Evans and S. Kielich (eds.), ibid, first edition (1992, 1993, 1997).

{9} M. W. Evans and J.-P. Vigier, “The Enigmatic Photon” (Kluwer, 1994 to 2002), in five volumes.

{10} M. W. Evans, papers on B(3) and ECE theory, Omnia Opera section of www.aias.us.